Symmetries, Clusters, and Synchronization Patterns in Complex Networks

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“O.K., let’s slowly lower in the grant money.”
Contributors and Co-Authors

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- Aaron Hagerstrom (Graduate Research Assistant, Physics)
Outline

• Synchronization of Dynamical Systems
• Describing Networks
  ▪ Master Stability Function
• Spatio-Temporal Optical Network
• Symmetries and Clusters
• Isolated Desynchronization
Synchronization in Nature

Synchronization in Engineered Systems

GPS

Power Grid

Chaotic Systems
Sensitivity to Initial Conditions

\[
\begin{align*}
\frac{dx_1}{dt} &= \sigma(y_1 - x_1) \\
\frac{dy_1}{dt} &= x_1(\rho - z_1) - y_1 \\
\frac{dz_1}{dt} &= x_1y_1 - \beta z_1
\end{align*}
\]

\[\sigma = 10, \ \rho = 28, \ \beta = 8/3.\]

\[x_1(0) = 1.0, \ y_1(0) = 1.0, \ z_1(0) = 1.0\]

\[x_1(0) = 1.001, \ y_1(0) = 1.0, \ z_1(0) = 1.0\]

Synchronization of Chaos

\[ \frac{dx_2}{dt} = \sigma(y_2 - x_2) + 1.5(x_1 - x_2) \]
\[ \frac{dy_2}{dt} = x_2(\rho - z_2) - y_2 \]
\[ \frac{dz_2}{dt} = x_2y_2 - \beta z_2 \]

\begin{align*}
  x_1(0) &= 1.0 \\
  y_1(0) &= 1.0 \\
  z_1(0) &= 1.0 \\
  x_2(0) &= 12.0 \\
  y_2(0) &= 1.0 \\
  z_2(0) &= 5.0
\end{align*}

\begin{align*}
  \text{slope} &= -\mu
\end{align*}
• Synchronization and Chaos
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C_{ij} = 1, if node i and j are connected
Assume all connections are identical, bidirectional
Generalizations:
- Weighted connections
- Directional links (C_{ij} \neq C_{ji})
Coupled Dynamical Systems

Continuous-time:

\[
\frac{d}{dt} x_i(t) = F(x_i(t)) + \sum_{j=1}^{N} C_{ij} H(x_j(t))
\]

Discrete-time:

\[
x_i[n + 1] = F(x_i[n]) + \sum_{j=1}^{N} C_{ij} H(x_j[n])
\]

Q1: **Can** these equations synchronize?
(Do they **admit** a synchronous solution \(x_1 = x_2 = \ldots x_N\)?)

Q2: **Do** these equations synchronize?
(... and is the synchronous solution **stable**?)
Synchronization of Coupled Systems

Laplacian Coupling Matrix (row sum = 0):

\[
C = \begin{bmatrix}
-4 & 1 & 1 & 1 & 1 & 1 \\
1 & -2 & 0 & 1 & 0 & \\
1 & 0 & -3 & 1 & 1 & \\
1 & 1 & 1 & -3 & 0 & \\
1 & 0 & 1 & 0 & -2 & \\
\end{bmatrix}
\]

\[x_1(t) = x_2(t) = \ldots = x_s(t), \quad \frac{d}{dt} x_s(t) = F(x_s(t))\]
Master Stability Function

Is the Synchronous Solution Stable

Master Stability Functions for Synchronized Coupled Systems

Louis M. Pecora and Thomas L. Carroll
Code 6343, Naval Research Laboratory, Washington, D.C. 20375
(Received 7 July 1997)

We show that many coupled oscillator array configurations considered in the literature can be put into a simple form so that determining the stability of the synchronous state can be done by a master stability function, which can be tailored to one’s choice of stability requirement. This solves, once and for all, the problem of synchronous stability for any linear coupling of that oscillator.

- Eigenvalues of $C$: \{0, $\lambda_1$, $\lambda_2$, $\lambda_3$, $\ldots$\}
- Stability condition: $M(\lambda_i) < 0$, for all $i$
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Electrooptic Feedback Loop

Map equation:

\[ x[n + 1] = a \sin^2(x) + \delta \]
Spatio-Temporal Optical Network

Video Feedback Network

spatial modulator

\(\lambda/4\) plate

PBS

laser / LED

camera

feedback network (FPGA)
Spatial Light Phase Modulator

- Same technology used in LCD displays
Coupled Dynamical Systems

- \( C_{ij} \) programmed through feedback (or by Fourier optics)
- SLM pixels are imaged onto camera pixels
- Almost arbitrary networks can be formed

- Coupled Map Equations:

\[
x_{i}[n + 1] = I(x_{i}[n]) + \sigma \sum_{j=1}^{N} C_{ij} I(x_{j}[n]) + \delta
\]

\( I(x) = a \sin^2(x) \)
Example: 11 node network
(6 links removed)

Network connections indicated by lines
Square patches of pixels for each node

Q: Can we predict and explain this cluster synchronization?
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Identifying Clusters and Symmetries

Group \( \mathcal{G} = \{ g_i \} \)

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
\end{pmatrix}
\]

Representation of the group \( \mathcal{G} \)
Each symmetry can be described by a N-dimensional permutation matrix $R_g$

The permutation matrix commutes with $C$: $R_g C = CR_g$

The equations of motion are invariant under symmetry operation

Orbits $=$ subsets of nodes that permute among themselves under symmetry group (clusters!)
Symmetries (Example)

- Symmetries and clusters are hard to identify in all but the simplest networks!
Hidden Symmetries

G.gens() = [(7,10), (6,7), (5,6), (4,8), (2,4)(8,9), (1,5), (1,11)]

0 symmetries

8640 symmetries


(Free) Tools for Computing Symmetries

- **GAP** = Groups, Algorithms, Programming (software for computational discrete algebra)
  http://www.gap-system.org/

- **Sage** = Unified interface to 100’s of open-source mathematical software packages, including GAP
  http://www.sagemath.org/

- **Python** = Open-source, multi-platform programming language
  http://www.python.org/
Example Output (GAP/Sage)

$G$.order(), $G$.gens() = 8640 [(9,10), (7,8), (6,9), (4,6), (3,7), (2,4), (2,11), (1,5)]

node sync vectors:
   Node 2

orb= [1, 5]
nodesyncvec  [0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0]
cyclesyncvec [1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0]

Node 1

orb= [2, 4, 11, 6, 9, 10]
nodesyncvec  [1, 0, 1, 0, 1, 0, 1, 0, 0, 1, 1]
cyclesyncvec [0, 1, 0, 1, 0, 1, 0, 0, 1, 1, 1]

Node 4

orb= [3, 7, 8]
nodesyncvec  [0, 0, 0, 1, 0, 0, 0, 1, 1, 0, 0]
cyclesyncvec [0, 0, 1, 0, 0, 1, 0, 1, 1, 0, 0]
Stability of Synchronization

linearizing about cluster states

- $C =$ coupling matrix in “node” coordinate system
- $T =$ unitary transformation matrix to convert to IRR coordinate system
- $B = TCT^{-1} =$ block-diagononalized form
Transformed Coordinate System
for perturbations away from synchrony

• $T$ is not an eigendecomposition or permutation matrix
• $T$ is found using irreducible representations (IRR) of symmetry group (computed from GAP)
Example: Diagonalization

Variational coupling matrix

\[
\begin{pmatrix}
-6.00 & -3.46 & 0.0 \\
-3.46 & -5.00 & -4.24 \\
0.0 & -4.24 & -6.00 \\
0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 \\
0.0 & 0.0 & 0.0 \\
\end{pmatrix}
\]

\[TCT^{-1}\]

Synchronization Manifold
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Cluster Synchronization in Experiment

- 11 nodes
- 49 links
- 32 symmetries
- 5 clusters:
  - Blue (2)
  - Red (2)
  - Green (2)
  - Magenta (4)
  - White (1)
• Pay attention to the magenta cluster:

\[ a = 0.7\pi \] \hspace{5cm} \[ a = 1.4\pi \]
Synchronization Error

![Graph showing RMS Synchronization Error over time]

- The graph illustrates the RMS Synchronization Error against time, with key points marked at $\frac{\pi}{2}$, $\pi$, $1.4\pi$, and $\frac{3\pi}{2}$.

- The error appears to increase significantly at $1.4\pi$, indicating a critical point in the synchronization process.

- The graph provides a visual representation of how synchronization errors evolve over time, which is crucial for understanding the dynamics of synchronized systems.
Intertwined Clusters

- Red and blue clusters are inter-dependent
- (sub-group decomposition)
Transverse Lyapunov Exponent
(linearizing about cluster synchrony)
Symmetries and Clusters in Random Networks

- N = 25 nodes (oscillators)
- 10,000 realizations of each type
- Calculate # of symmetries, clusters

Random

\( n_{\text{delete}} = 20 \)

Scale-free Tree

Scale-free \( \gamma \)


Symmetries, clusters and subgroup decompositions seem to be universal across many network models.
Power Network of Nepal
Mesa Del Sol Electrical Network

- 4096 symmetries
- 132 Nodes
- 20 clusters
- 90 trivial clusters
- 10 subgroups
Symmetries & Clusters in Larger Networks


<table>
<thead>
<tr>
<th>Network</th>
<th>(N_g)</th>
<th>(M_g)</th>
<th>(a_g)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Human B Cell Genetic Interactions</td>
<td>5,930</td>
<td>64,645</td>
<td>(5.9374 \times 10^{13})</td>
</tr>
<tr>
<td><em>C. elegans</em> Genetic Interactions</td>
<td>2,060</td>
<td>18,000</td>
<td>(6.9985 \times 10^{161})</td>
</tr>
<tr>
<td>BioGRID datasets</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Human</td>
<td>7,013</td>
<td>20,587</td>
<td>(1.2607 \times 10^{485})</td>
</tr>
<tr>
<td><em>S. cerevisiae</em></td>
<td>5,295</td>
<td>50,723</td>
<td>(6.8622 \times 10^{64})</td>
</tr>
<tr>
<td><em>Drosophila</em></td>
<td>7,371</td>
<td>25,043</td>
<td>(3.0687 \times 10^{493})</td>
</tr>
<tr>
<td><em>Mus musculus</em></td>
<td>209</td>
<td>393</td>
<td>(5.3481 \times 10^{125})</td>
</tr>
<tr>
<td>Internet (Autonomous Systems Level)</td>
<td>22,332</td>
<td>45,392</td>
<td>(1.2822 \times 10^{11,298})</td>
</tr>
<tr>
<td>US Power Grid</td>
<td>4,941</td>
<td>6,594</td>
<td>(5.1851 \times 10^{152})</td>
</tr>
</tbody>
</table>

> 88% of nodes are in clusters in all above networks
Summary

• Synchronization is a widespread in both natural and engineered systems
• Many systems exhibit patterns or clusters of synchrony
• Synchronization patterns are intimately connected to the hidden symmetries of the network
For more information:

- L. M. Pecora, F. Sorrentino, A. M. Hagerstrom, TEM, and R. Roy
  “Cluster synchronization and isolated desynchronization in complex networks with symmetries”

- B. Ravoori, A. B. Cohen, J. Sun, A. E. Motter, TEM, and R. Roy,
  “Robustness of Optimal Synchronization in Real Networks”

- A. B. Cohen, B. Ravoori, F. Sorrentino, TEM, E. Ott and R. Roy,
  “Dynamic synchronization of a time-evolving optical network of chaotic oscillators”

- TEM, A. B. Cohen, B. Ravoori, K. R. B. Schmitt, A. V. Setty, F. Sorrentino, C. R. S. Williams,
  E. Ott and R. Roy,
  “Chaotic Dynamics and Synchronization of Delayed-Feedback Nonlinear Oscillators”

  “Adaptive synchronization of coupled chaotic oscillators”