Asymptotically Free Gauge Theories. I*

David J. Gross†
National Accelerator Laboratory, P. O. Box 500, Batavia, Illinois 60510
and Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540

Frank Wilczek
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540
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Asymptotically free gauge theories of the strong interactions are constructed and analyzed. The reasons for doing this are recounted, including a review of renormalization-group techniques and their application to scaling phenomena. The renormalization-group equations are derived for Yang-Mills theories. The parameters that enter into the equations are calculated to lowest order and it is shown that these theories are asymptotically free. More specifically the effective coupling constant, which determines the ultraviolet behavior of the theory, vanishes for large spacelike momenta. Fermions are incorporated and the construction of realistic models is discussed. We propose that the strong interactions be mediated by a "color" gauge group which commutes with SU(3) × SU(3). The problem of symmetry breaking is discussed. It appears likely that this would have a dynamical origin. It is suggested that the gauge symmetry might not be broken and that the severe infrared singularities prevent the occurrence of noncolor singlet physical states. The deep-inelastic structure functions, as well as the electron-positron total annihilation cross section are analyzed. Scaling obtains up to calculable logarithmic corrections, and the naive light-cone or parton-model results follow. The problems of incorporating scalar mesons and breaking the symmetry by the Higgs mechanism are explained in detail.

I. INTRODUCTION

In recent years the renormalization group has played an increasingly important role in the study of the asymptotic behavior of renormalizable field theories. This approach has acquired new importance due to the recent discovery that non-Abelian gauge theories are asymptotically free.[1-3] In this paper we shall amplify and extend the results reported in Ref. 1.

The renormalization group dates from the fundamental work of Gell-Mann and Low,[4] who studied the asymptotic behavior of the photon propagator in quantum electrodynamics. The remarkable discovery of Gell-Mann and Low was that the asymptotic form of the photon propagator was determined by the zeros of a certain, calculable, function of the coupling constant and not by the actual value of the charge. The renormalization-group equations were extended by Bogolubov and Shirkov to the vertex function[5] and employed to analyze the ultraviolet and infrared behavior of quantum electrodynamics and other field theories.[6,7] (For a review of this work see Ref. 8.)

The basic idea underlying the renormalization-group equations is very simple. A renormalizable field theory contains two types of parameters—masses or coupling constants with positive dimensions of mass (i.e., due to $M\bar{\psi}\psi$ or $\lambda\phi^3$ terms in the Lagrangian) and dimensionless coupling constants (i.e., due to $\lambda\phi^4$ or $\bar{\psi}\lambda^\mu\psi A_\mu$ terms in the Lagrangian). Coupling constants with negative dimensions of mass give nonrenormalizable theories. If one considers a Green's function for large and spacelike momenta (so as to exclude any Landau singularities), then one would expect that the generalized mass terms in the Lagrangian ($M\bar{\psi}\psi$ or $\lambda\phi^3$) could be neglected. In other words the leading asymptotic behavior of the Green's functions should be the same as would be calculated in a massless theory. This can be proved, to any finite order in perturbation theory, by using Weinberg's theorem.[9] The massless theory contains no dimensional parameters to set the scale of momenta, therefore one might expect that the asymptotic behavior of the amplitudes would be determined by pure dimensional analysis. This is called naive or cannonical scaling. It does not occur in practice, since the massless theory does contain a hidden dimensional parameter. This parameter, μ, must be introduced in order to perform the subtractions necessary to renormalize the theory and render it finite. Due to infrared singularities these subtractions, for the massless theory, must be performed off shell, say at some spacelike momenta $p^2=-\mu^2$. The subtractions then define the physical coupling constants and the scale of the fields (which are determined by the wave-function renormalization constants). The subtraction point, μ, is arbitrary. If we change the subtraction point the net effect is to change the value of the coupling constants and the scale of the fields. This fact is expressed by the renormalization-group equations.

A change in the subtraction point, μ, is equivalent to a change in the scale of all momenta since μ is the sole parameter that fixes the momentum scale. Therefore one can use the renormalization group to relate Green's functions for one set of momenta and coupling constant to Green's function for a scaled set of momenta and different values of the coupling constants. To do this one only needs to know the functions which determine the change in coupling constant and scale of fields due to a change in μ. In particular the asymptotic form of the amplitudes (for large spacelike momenta) can be related to amplitudes at some fixed momenta and an effective coupling constant (which would be determined by performing the subtractions at asymptotic values of the momenta). The asymptotic value of this effective coupling constant will determine the ultraviolet behavior of the theory. It is given by the zeros of a calculable function, which are called fixed points of the renormalization group.

For about ten years there was little interest in the renormalization group.[10] This was probably due to the following reasons. First, the renormalization group provided information about Green's functions for large spacelike momenta, which are of no direct physical interest. This approach is much less informative about Minkowski momenta and on-shell amplitudes—since we lack an extension to this region of Weinberg's powerful theorem.[9] Second, it was soon discovered, in all the cases investigated at the time, that the ultraviolet behavior was not calculable using perturbation theory. Thus it appeared that the renormalization group provided a framework in which one could discuss, but not calculate, the asymptotic behavior of amplitudes in a physically uninteresting region.

This situation has changed in the last few years due to the following developments. First there was an increased interest in the matrix elements of local currents at short distances[11] (or their Fourier transforms for large spacelike momenta). This interest was further increased by the advent of new experiments on deep-inelastic lepton-hadron scattering at SLAC, and by Bjorken's prediction and the experimental indications of scaling.[12] The theoretical framework for the discussion of products of currents at short distances was provided by Wilson's operator-product expansion.[13] Although Wilson had emphasized that, in general, one does not expect naive scaling, the experimental indications of Bjorken scaling motivated the development of phenomenological scaling models. One approach was to abstract from free-field theory, or interacting-field theories with an ultraviolet cutoff, the short-distance structure of current products,[14] the other was to hypothesize the existence of pointlike constituents of hadrons (partons).[15] Assuming a quark field-theoretic model, or that the partons had the quantum numbers of quarks, many relations and sum rules were derived. At present the experimental data are consistent with spin-$\frac{1}{2}$ (Ref. 16) quarklike constituents. All attempts to provide dynamical explanations for these models were unsuccessful.

Meanwhile significant developments of the renormalization group techniques were made. The equations that determine the change in momentum scale of massive field theories were derived by Callan[17] and Symanzik.[18] These equations, for large spacelike momenta, reduce to the differential form of the renormalization-group equations already derived by Orsiannikov in a little-noticed paper in 1956.[19] In addition it was realized that the renormalization-group approach is the key to asymptotic behavior of the coefficient functions in Wilson's operator-product expansion, and thus the related behavior of electroproduction structure functions in the Bjorken region.[20-23]

In particular it was realized that Bjorken scaling could be understood within the framework of the renormalization group if there was an ultraviolet-(UV) stable fixed point of the renormalization group. At the fixed point, however, the anomalous dimensions of the relevant operators in Wilson's expansion would all have to vanish. All the indications are that this can only occur if the value of the fixed point of the renormalization group is zero.[24,25] This has recently been proved in a large class of field theories.[26] In such a theory the effective coupling constant vanishes for large spacelike momenta and we describe this phenomena by saying that the theory is *asymptotically free*. An asymptotically free theory will exhibit Bjorken scaling (up to, perhaps, logarithmic corrections) and in addition will lead to all the naive light-cone- or parton-model results.[26]

The possibility that a given field theory is asymptotically free is easily explored by simple perturbation-theory calculations. Quantum electrodynamics was known, from the original work of Gell-Mann and Low, not to be asymptotically free. Zee extended this result to scalar-fermion theories involving one coupling constant.[27] Recently Coleman and one of us (D.J.G.) have proved that *no theory which does not involve non-Abelian gauge mesons can be asymptotically free.*[28] Together with the recent discovery that non-Abelian gauge theories are asymptotically free[1-3] these developments provide a compelling case for a non-Abelian gauge theory of the strong interactions. Indeed *if* one accepts the renormalization-group approach and the experimental reality of Bjorken scaling as an asymptotic phenomenon then there is, probably, no other choice.[29] This possibility is explored in the following.

In Sec. II we shall outline the derivation of the renormalization-group equations for pure Yang-Mills theories (which involve only gauge fields). These equations are discussed in greater detail in Appendix A. We also discuss the notion of the effective coupling constant and exhibit the solution of the renormalization-group equations.

In Sec. III we calculate the renormalization-group parameters for pure Yang-Mills theories.

In Sec. IV we incorporate fermions into the gauge theories, without destroying their asymptotic freedom. The large-momentum behavior of the effective coupling constant and Green's functions is derived.

In Sec. V the construction of realistic physical models of the strong interactions is discussed. We analyze the structure functions of deep-inelastic scattering and the total electron-positron annihilation cross section in these models. These applications will be explored further in a forthcoming publication.[30] The major problem remaining in these gauge theories is how to break the gauge symmetry and provide masses for the vector mesons. Various dynamical possibilities are discussed in Sec. V.

In Sec. VI we incorporate scalar mesons into asymptotically free gauge theories. The difficulties encountered in achieving this are described, and the failure to construct models in which the Higgs mechanism generates masses for all the vector mesons is explained.

Section VII contains some concluding remarks.

II. THE RENORMALIZATION-GROUP EQUATIONS FOR YANG-MILLS THEORIES

In this section we shall derive the renormalization-group equations for a pure Yang-Mills theory. The only restriction on the form of the theory will be the requirement that the gauge group be non-Abelian and semisimple.

The classical Yang-Mills Lagrangian density is

$$L = -\tfrac{1}{2}\mathrm{Tr}\{(\partial_\mu B_\nu - \partial_\nu B_\mu - g[B_\mu B_\nu])^2\}, \qquad (2.1)$$

where

$$B_\mu(x) = B_\mu^a(x) t_a \qquad (2.2)$$

is a matrix of Hermitian vector fields (summation over repeated indices is implied). The matrices t_a generate a semisimple Lie group G:

$$[t_a, t_b] = i C_{abc} t_c \qquad (2.3)$$

and are normalized according to

$$\mathrm{Tr}\{t_a t_b\} = \tfrac{1}{2}\delta_{ab}\,. \qquad (2.4)$$

This Lagrangian is singular due to its invariance under the gauge group. Therefore a proper quantization of (2.1) necessitates the addition of a gauge-fixing term to the Lagrangian, say $-(1/\alpha)\mathrm{Tr}\{(\partial^\mu B_\mu)^2\}$. The presence of this term then requires the appearance of Feynman-Faddeev-Popov ghosts.[31] The net result is that the effective Lagrangian used to derive the Feynman rules is

$$\begin{aligned} L = &-\tfrac{1}{2}\mathrm{Tr}\{(\partial_\mu B_\nu - \partial_\nu B_\mu - g[B_\mu B_\nu])^2\} \\ &-\frac{1}{\alpha}\mathrm{Tr}\{(\partial^\mu B_\mu)^2\} \\ &+2\,\mathrm{Tr}\{\partial_\mu\phi^*\partial_\mu\phi - g\partial_\mu\phi^*[B_\mu,\phi]\}, \end{aligned} \qquad (2.5)$$

where $\phi = \phi^a t_a$ is a massless, complex, scalar field which propagates in closed loops only and obeys Fermi statistics. The resulting Feynman rules are summarized in Fig. 1.

Due to the presence of massless particles and the resulting infrared singularities it would appear that the S matrix does not exist, at least in perturbation theory, for the pure Yang-Mills theory. One can however consider the off-shell Green's function for such theories, at all but exceptional momenta.[32] In the following, this restriction is always to be understood. The large-momentum behavior of such Green's functions is not without physical interest since it bears directly on the ultraviolet behavior of more realistic models with symmetry-breaking and mass terms.

Up till now we have discussed the Yang-Mills

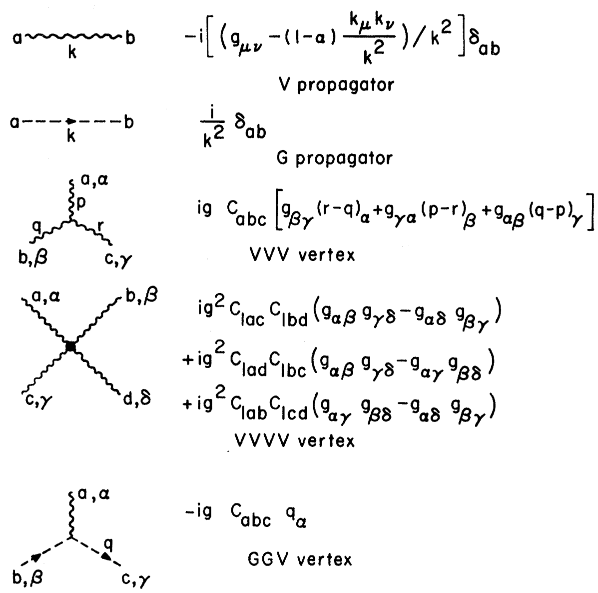


FIG. 1. The Feynman rules for a pure Yang-Mills theory (all momenta flow into the vertices).

theory formally, that is without regard to the necessary renormalization procedure. There exist two ways of regularizing these theories. The most elegant method is that of dimensional regularization as discussed by 't Hooft and Veltman.[33] Alternatively, one can add gauge-invariant higher-derivative terms to the Lagrangian, following Slavnov[34] and Lee and Zinn-Justin.[35] Both of these methods have the necessary virtue of maintaining explicit gauge invariance.

With such a regularization the only primitively divergent amplitudes are the vector-meson and ghost two- and three-point functions and the vector-meson four-point function. The necessary subtractions are severely restricted by the Ward identities, which render all divergences logarithmic and relate the various divergent amplitudes. This is merely a reflection of the gauge invariance which limits the renormalized couplings to those displayed in the Lagrangian (2.5).[36]

The theory is then determined by specifying the subtraction constants at some convenient subtraction point. It is, of course, impossible to subtract at zero four-momentum, due to the infrared singularities. One therefore performs the subtraction at an (arbitrary) Euclidean point $p^2 = -\mu^2$.

We define the vector wave-function renormalization constant Z_3 in terms of the unrenormalized transverse vector propagator:

$$D_{\text{un}}^{\text{tr}}(k)_{\mu\nu}^{ab}\big|_{k^2=-\mu^2} = \frac{i}{\mu^2} Z_3\left(g_{\mu\nu} + \frac{k_\mu k_\nu}{\mu^2}\right)\delta_{ab}\,; \qquad (2.6)$$

the ghost wave-function renormalization constant $\tilde{Z}_3$ in terms of the unrenormalized ghost propagator:

$$G_{\text{un}}^{ab}(-\mu^2) = -\frac{i}{\mu^2}\tilde{Z}_3\delta_{ab}\,; \qquad (2.7)$$

the vector charge renormalization constant Z_1 in terms of the unrenormalized vector three-point vertex:

$$\Gamma_{\text{un}}^{abc}(p,q,r)_{\mu\nu\lambda}\big|_{p^2=q^2=r^2=-\mu^2} = Z_1^{-1}\Gamma_{\text{BARE}}^{abc}(p,q,r)_{\mu\nu\lambda}\,; \qquad (2.8)$$

and the ghost-vector charge renormalization constant $\tilde{Z}_1$ in terms of the unrenormalized ghost-vector three-point vertex:

$$\Gamma_{\text{un}}^{a;b,c}(p;q,r)_\mu\big|_{p^2=q^2=r^2=-\mu^2} = \tilde{Z}_1^{-1}\Gamma_{\text{BARE}}^{a;b,c}(p;q,r)_\mu\,. \qquad (2.9)$$

The Ward identities[34,35] ensure that these constants are related by

$$Z_3/Z_1 = \tilde{Z}_3/\tilde{Z}_1 \qquad (2.10)$$

and that the longitudinal part of the inverse vector propagator is unrenormalized:

$$D_{\text{un}}^{-1}(k)_{\mu\nu}^{ab} = \cdots + \frac{i}{\alpha_u}k_\mu k_\nu\,, \qquad (2.11)$$

where α_u is the unrenormalized gauge parameter.

The renormalized Green's functions are then defined by scaling the fields according to

$$(B_\mu^a)_r = Z_3^{-1/2}(B_\mu^a)_u\,,\qquad (\phi^a)_r = \tilde{Z}_3^{-1/2}(\phi^a)_u\,, \qquad (2.12)$$

and defining the renormalized charge to be

$$g_r = Z_3^{+3/2}Z_1^{-1}g_u \qquad (2.13)$$

and the renormalized gauge parameter to be

$$\alpha_r = Z_3^{-1}\alpha_u\,. \qquad (2.14)$$

The renormalized one-particle irreducible (1PI) Green's functions $\Gamma^{(n)}_{\mu_1\cdots\mu_n}(P_1\cdots P_n)$ [$\Gamma^{(2)}$ is the inverse propagator],

$$\Gamma^{(n)}_{\mu_1\cdots\mu_n}(P_1\cdots P_n) = Z_3^{n/2}\,\Gamma^{(n)}_{\mu_1\cdots\mu_n}(P_1\cdots P_n)_{\text{unrenormalized}}\,, \qquad (2.15)$$

are then finite functions of the renormalized charge g, the gauge parameter α, and the renormalization point μ. (When g and α appear without subscripts they will refer to their renormalized values.)

The choice of renormalization point, μ, is arbitrary. Any change in μ can be compensated by a corresponding change of the charge, the scale of the fields, and the gauge parameter. The renormalization-group equations reflect this fact. These equations are most simply derived[37] by noting that the unrenormalized 1PI Green's functions $\Gamma_u^{(n)}(\Lambda; g_u)$ (we suppress the momenta and vector index labels of these 1PI Green's functions), when expressed as functions of the cutoff, the bare coupling constant g_u, and the gauge parameter α_u, are independent of μ:

$$\mu\frac{\partial}{\partial\mu}\Gamma_u^{(n)}(\Lambda, g_u, \alpha_u) = 0\,. \qquad (2.16)$$

Using Eq. (2.15) and the chain differentiation rule we have

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g,\alpha)\frac{\alpha}{\alpha g} - n\gamma(g,\alpha) + \delta(g,\alpha)\frac{\partial}{\partial\alpha}\right] \times \Gamma^{(n)}(g,\alpha,\mu) = 0\,, \qquad (2.17)$$

where

$$\beta(g,\alpha) = \mu\frac{\partial g}{\partial\mu}\bigg|_{g_\mu,\alpha_u,\Lambda\ \text{fixed}}\,, \qquad (2.18)$$

$$\gamma(g,\alpha) = \tfrac{1}{2}\mu\frac{\partial \ln Z_3}{\partial\mu}\bigg|_{g_\mu,\alpha_u,\Lambda\ \text{fixed}}\,, \qquad (2.19)$$

$$\delta(g,\alpha)=\mu\frac{\partial\alpha}{\partial\mu}\bigg|_{g_\mu,\alpha_u,\Lambda\text{ fixed}}. \tag{2.20}$$

The fact that β, γ, δ are finite functions of g and α is an immediate consequence of (2.17) and the fact that $\Gamma^{(n)}$ is finite when expressed in terms of the renormalized parameters.

The last term in the renormalization-group equation is peculiar to gauge theories. As a consequence of Eq. (2.14) we have that

$$\begin{aligned}\delta(g,\alpha)&=-\frac{\alpha_u}{Z_3^{\,2}}\mu\frac{\partial Z_3}{\partial\mu}\bigg|_{g_\mu,\alpha_u,\Lambda\text{ fixed}}\\&=-2\alpha\gamma(g,\alpha)\ .\end{aligned} \tag{2.21}$$

The renormalization-group equations assume a particularly simple form in Landau gauge where $\alpha_u=\alpha=0$, for in that gauge $\delta=0$. This is a reflection of the fact that the longitudinal part of the vector propagator vanishes in this gauge, and a change in renormalization point does not change the gauge parameter. In the following we shall often restrict ourselves to this gauge. Ultimately we are interested in the physical consequences of these theories which are determined by gauge-invariant amplitudes. For such amplitudes the change in gauge parameter can be reabsorbed by a change in coupling and scale of fields. This is explained in some detail in Appendix A, where we also show that the lowest-order terms (of order g^3) in $\beta(g,\alpha)$ are independent of α.

The utility of the renormalization-group equations is that they determine the change in the Green's functions as we scale all momenta uniformly. Consider the 1PI amplitudes

$$\Gamma^{(n)}(\lambda p_1,\lambda p_2,\ldots,\lambda p_n;g,\mu)$$

(in Landau gauge), where p_i is some set of nonexceptional Euclidean momenta, and λ is a nonvanishing parameter. Pure dimensional analysis implies that

$$\Gamma^{(n)}(\lambda p_i,\ldots,\lambda p_n;g,\mu)=\mu^{4-n}\Gamma^{(n)}\left(\frac{\lambda p_1}{\mu},\ldots,\frac{\lambda p_n}{\mu}\right) \tag{2.22}$$

so that (2.17) can be rewritten as

$$\left[\lambda\frac{\partial}{\partial\lambda}-\beta(g)\frac{\partial}{\partial g}-4+n[1+\gamma(g)]\right]\times\Gamma^{(n)}(\lambda p_1,\ldots,\lambda p_n;g,\mu)=0. \tag{2.23}$$

The general solution of (2.23) is most transparently expressed in terms of the *effective coupling constant* $\bar{g}(t,g)$:

$$\begin{aligned}&t=\ln\lambda\ ,\\&\frac{d}{dt}\bar{g}(t,g)=\beta(\bar{g}),\quad \bar{g}(0,g)=g\ .\end{aligned} \tag{2.24}$$

This function is given implicitly by

$$\int_g^{\bar{g}}\frac{dx}{\beta(x)}=t \tag{2.25}$$

and satisfies

$$\left[\frac{\partial}{\partial t}-\beta(g)\frac{\partial}{\partial g}\right]\bar{g}(t,g)=0\ . \tag{2.26}$$

The physical meaning of the effective coupling constant $\bar{g}(t,g)$ is that it equals the renormalized coupling constant defined by performing the subtractions indicated in (2.6)–(2.9) at the Euclidean point

$$p^2=-\mu^2\lambda^2=-\mu^2e^{2t}\ .$$

It is expressed in terms of the renormalized coupling constant g (which was determined by subtracting at $p^2=-\mu^2$). The renormalization-group equation (2.24) then determines the effect on $\bar{g}$ of a change in the subtraction point.

In terms of $\bar{g}$ we have

$$\Gamma^{(n)}(\lambda p_1,\ldots,\lambda p_n;g,\mu)=\Gamma^{(n)}(p_1,\ldots,p_n;\bar{g}(t,g),\mu)\lambda^{4-n}\times\exp\left(-n\int_0^t dx\,\gamma(\bar{g}(x,g))\right). \tag{2.27}$$

Of particular interest is the large-λ limit of this solution, for it determines the ultraviolet behavior of the Green's functions even in the presence of mass terms (which here must arise from symmetry breaking). This limit will in turn be controlled by the large-t behavior of the effective coupling constant $\bar{g}(t,g)$. If Eq. (2.4) admits a solution such that

$$\lim_{t\to\infty}\bar{g}(t,g)=g_\infty\ , \tag{2.28}$$

then we say that g_∞ is an ultraviolet-stable fixed point. The asymptotic behavior of $\Gamma^{(n)}$ is then controlled by g_∞ according to

$$\begin{aligned}&\Gamma^{(n)}(\lambda p_1,\ldots,\lambda p_n,g,\mu)\\&\quad\to\Gamma^{(n)}(p_1,\ldots,p_n;g_\infty,\mu)\lambda^{4-n-n\gamma(g_\infty)}\\&\qquad\times\exp\left(-n\int_0^{\ln\lambda}[\gamma(\bar{g}(x,g))-\gamma(g_\infty)]\,dx\right)\end{aligned} \tag{2.29}$$

so that $\gamma(g_\infty)$ is the anomalous dimension of the field.

The fixed points of the renormalization group are determined by the zeros of $\beta(g)$; i.e., $\beta(g_\infty)=0$. However, not all such zeros are UV-stable. Thus if β has a simple zero at g_∞ this will be UV-stable if and only if

$$\beta(g_\infty)=0;\quad \frac{d}{dg}\beta(g_\infty)<0\ . \tag{2.30}$$

A zero of β at which $d\beta/dg>0$ is said to be an infrared- (IR) stable fixed point, since $\bar{g}$ approaches such a fixed point when $\lambda\to 0$ $(t\to-\infty)$.

A theory is said to be *asymptotically free* if g_∞ vanishes. In that case $\gamma(g_\infty)=0$ and the Green's functions can be expanding for large Euclidean momenta, in a asymptotic series in $\bar{g}(t)$ [plus a modification due to the integral in (2.29)]. Since in all theories $\beta(0)=0$, the origin of coupling-constant space is either UV- or IR-stable. It has recently been proved[28] that no renormalizable field theory without non-Abelian gauge fields can be asymptotically free.

Asymptotically free field theories are clearly of great theoretical interest. They provide one with models in which the asymptotic behavior of amplitudes is calculable by ordinary perturbation theory. In addition there appears to be evidence, experimental and theoretical, that such theories are required to explain deep-inelastic scattering. The phenomenon of scaling predicted by Bjorken[12] is, up to logarithmic corrections, a true asymptotic feature of asymptotically free theories. Furthermore, it now appears that Bjorken scaling can only occur if the strong interactions are asymptotically free.[26] The fact that the only theories that can be asymptotically free are those involving non-Abelian gauge fields and that, as we shall see in the following, asymptotically free gauge theories can be constructed, is a strong argument for a gauge theory of the strong interactions.

III. CALCULATION OF THE RENORMALIZATION-GROUP PARAMETERS

We shall now proceed to calculate the renormalization-group parameters β and γ to lowest nontrivial order in perturbation theory, that is to order g^3 and g^2, respectively. To calculate these functions one must calculate the renormalization constants Z_3 and Z_1 to order g^2 and use Eqs. (2.18) and (2.19). These constants can only depend on the renormalization point μ via the ratio Λ/μ, where Λ is the ultraviolet cutoff. It is therefore sufficient to calculate the logarithmically divergent terms (of order g^2) in Z_3 and Z_1. It then follows that

$$\beta(g,\alpha)=-g\frac{\partial}{\partial\ln\Lambda}\left(\frac{Z_3{}^{3/2}}{Z_1}\right), \tag{3.1}$$

$$\gamma_V(g,\alpha)=-\frac{1}{2}\frac{\partial}{\partial\ln\Lambda}\ln Z_3 . \tag{3.2}$$

(We denote the anomalous dimension of the vector mesons by γ_V.) The calculations are thus greatly simplified. In particular it is not necessary to specify the regularization method employed.

The wave-function renormalization constant is determined by the vector-meson self-energy graphs shown in Fig. 2. We calculate from these graphs and (2.6) that

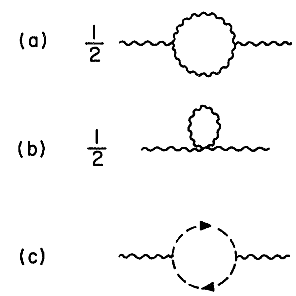


FIG. 2. The vector-meson self-energy graphs.

$$Z_3=1+\frac{g^2}{16\pi^2}\left(\tfrac{13}{3}-\alpha\right)C_2(G)\ln\Lambda, \tag{3.3}$$

where $C_2(G)$ is the value of the quadratic Casimir operator for the adjoint representation of the gauge group G. Namely,

$$\sum_{c,d}C_{acd}\,C_{bcd}=C_2(G)\delta_{ab}; \tag{3.4}$$

in the case of SU(N), $C_2(\mathrm{SU}(N))=N$.

Similarly the charge renormalization constant is determined by the Feynman graphs shown in Fig. 3 and yields

$$Z_1=1+\frac{g^2}{16\pi^2}\left(\frac{17}{6}-\frac{3\alpha}{2}\right)C_2(G)\ln\Lambda . \tag{3.5}$$

A useful check on the above calculations is provided by the Ward identity, Eq. (2.10), which relates the vector-meson and the ghost-renormalization constants. The latter are much easier to calculate. The relevant graphs are shown in Figs. 4

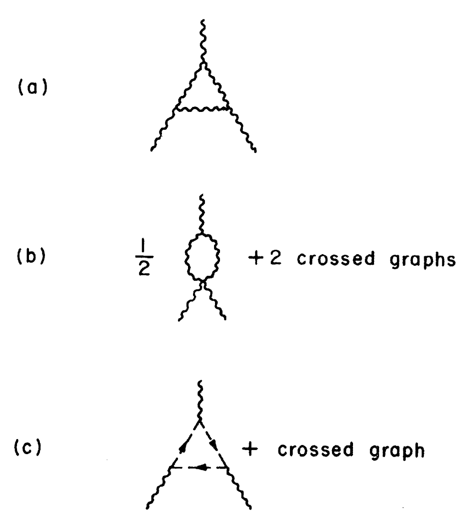


FIG. 3. The trilinear vector-meson vertex corrections.

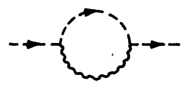


FIG. 4. The ghost self-energy graph.

and 5 and result in

$$\tilde{Z}_3 = 1 + \frac{g^2}{16\pi^2}\left(\frac{3}{2} - \frac{\alpha}{2}\right) C_2(G) \ln\Lambda , \tag{3.6}$$

$$\tilde{Z}_1 = 1 + \frac{g^2}{16\pi^2}(-\alpha) C_2(G) \ln\Lambda . \tag{3.7}$$

The calculation of β and γ_V is then mere arithmetic. The result is

$$\beta(g, \alpha) = -\frac{g^3}{16\pi^2}\tfrac{11}{3} C_2(G) + O(g^5) \tag{3.8}$$

$$\gamma_V(g, \alpha) = -\frac{g^2}{16\pi^2}(\tfrac{13}{3} - \alpha) C_2(G). \tag{3.9}$$

It is obvious from the above that for these gauge theories the origin is UV-stable. If g^2 is small enough then the solution to (2.24) will approach zero, as $1/t$ for large t. We shall explore the consequences of this following the incorporation of fermions. We cannot provide a deeper understanding of why non-Abelian gauge theories are so different from all other field theories in this respect. Perhaps the serious infrared singularities of Yang-Mills theories are reflected in the IR instability, and thus the UV stability, of the origin.

IV. INCORPORATION OF FERMIONS

We now consider a gauge theory which includes matter fields. Fermions are easily incorporated without introducing any new coupling constants, and without destroying the asymptotic freedom. We add fermions by adding to the Lagrangian the term

$$L_F = \bar{\psi}(i\partial\!\!\!/ - M - g\sigma^a \cdot B\!\!\!/^a)\psi , \tag{4.1}$$

where the σ^a are the matrices of the representation R of the gauge group G according to which ψ transforms. The fermions may have nonvanishing, but symmetric, masses since we are not considering here chiral gauge groups.

In the presence of a fermion mass the renormalization-group equations are no longer valid. Instead one can derive the Callan-Symanzik equations[17,18] which contain, in addition to the renormalization-group operator, an inhomogeneous term arising from mass insertions. The renormalization-group parameters are unaffected by these mass terms. For large Euclidean momenta the Callan-Symanzik equations reduce to the renormalization-group equations. This is a consequence, to any finite order in perturbation theory, of Weinberg's theorem. More precisely consider the large-λ limit of a Green's function $\Gamma^{(n)}(\lambda p_i)$ for nonexceptional Euclidean momenta. The leading power in λ of $\Gamma^{(n)}$ to any finite order in perturbation theory is denoted by $\Gamma^{(n)}_{\text{ASY}}(\lambda p_i)$ and satisfies the renormalization-group equation [e.g., Eq. (2.23)]. We should emphasize that an important assumption is being made here, namely, that the leading-power behavior of perturbation theory is identical with that of the actual solution.

The effect of the fermions on β and γ is easily calculated. The fermions contribute to Z_3 via the graph shown in Fig. 6 and to Z_1 via the graph shown in Fig. 7. These yield the contributions

$$Z_3^F = Z_1^F = -\frac{g^2}{8\pi^2}\tfrac{4}{3} T(R) \ln\Lambda , \tag{4.2}$$

where $T(R)$ is defined by

$$\text{Tr}(\sigma^a\sigma^b) = T(R)\delta_{ab} . \tag{4.3}$$

We note the elementary identity between the value of the Casimir operator for the representation R and $T(R)$ given by

$$rT(R) = d(R)C_2(R) , \tag{4.4}$$

where $d(R)$ is the dimension of the representation R and r the dimension (number of generators) of the group. For example, in the case of SU(N) we have for the vector representation (N)

$$T((N)) = \tfrac{1}{2},$$

$$C_2((N)) = \frac{N^2 - 1}{2N} ,$$

whereas for the adjoint representation $T(\text{ADJ}) = N = C_2(G)$.

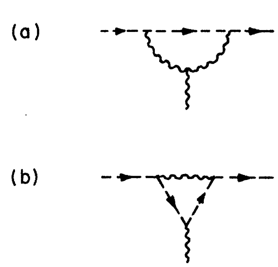


FIG. 5. The ghost–ghost–vector-meson vertex corrections.

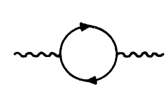


FIG. 6. The contribution of the fermions to the vector-meson self-energy.

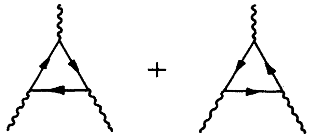


FIG. 7. The contribution of the fermions to the trilinear vector-meson coupling.

The equality of Z_3^F and Z_1^F is an immediate consequence of the Ward identity, Eq. (2.10), and the fact that $\tilde{Z}_1$ and $\tilde{Z}_3$ receive no contributions from fermion loops to lowest order. The resulting contributions to $\beta(g)$ and $\gamma(g)$ are

$$\beta_F(g) = +\frac{g^3}{16\pi^2}\,\frac{4}{3}\,T(R)\ , \tag{4.5}$$

$$\gamma_V^{(F)}(g) = +\frac{g^2}{16\pi^2}\,\frac{8}{3}\,T(R)\ . \tag{4.6}$$

These contributions are opposite in sign to those arising from the vector-meson loops. Thus the fermions tend to destabilize the origin. However, there is room to spare. As long as

$$T(R) < \tfrac{11}{4}\,C_2(G) \tag{4.7}$$

the theory will be asymptotically free. This requirement is not very restrictive as to the number of fermions allowed in the theory. For example if the gauge group is SU(3) ($C_2 = 3$), one can accommodate as many as 16 triplets or 2 octets of fermions without losing asymptotic freedom.

It is therefore possible to construct a large class of theories in which the renormalization-group equations for the effective coupling constant $\bar{g}(t,g)$ take the form

$$\frac{d}{dt}\bar{g}^2 = 2\bar{g}\beta(\bar{g}) = -b_0\bar{g}^4 + b_1\bar{g}^6 + \cdots\ , \tag{4.8}$$

where

$$b_0 = \frac{1}{8\pi^2}\left[\tfrac{11}{3}\,C_2(G) - \tfrac{4}{3}\,T(R)\right]. \tag{4.9}$$

The structure of this equation assures us that as long as $\bar{g}(0,g)$ is small enough then for large t we will have

$$\bar{g}^2(t,g) \underset{t\to\infty}{\approx} b_0^{-1}t^{-1} + b_1 b_0^{-3} t^{-2}\ln t + O(1/t^2)\ . \tag{4.10}$$

This will be the true asymptotic behavior of the effective coupling constant as long as $\bar{g}(0,g) = g$ is in the domain of attraction of the fixed point $\bar{g} = 0$.[38] The size of this domain is determined by the value of the first zero of $\beta(g)$, namely, if

$$\beta(g) = 0 \quad \text{for } g = g_1\ , \tag{4.11}$$

then the domain of attraction of the origin is given by[39]

$$0 < g^2 < g_1^2\ . \tag{4.12}$$

Perturbation theory of course tells us very little about the nonvanishing zeros of β. If we calculate β to any finite order, we might expect zeros to occur at values

$$\frac{\bar{g}^2}{4\pi}\,C_2(G) \approx O(1) \tag{4.13}$$

since this is the effective expansion parameter. On the other hand such approximations to β are totally unreliable for these values of g. It is perfectly possible that in gauge theories β is negative semidefinite and all values of the coupling constant are in the domain of attraction of the origin. Clearly for such theories to describe the strong interactions it is necessary that this domain be relatively large. It would therefore be useful to know the value of b_1, which we have not calculated.

The value of b_1 would be interesting in another context. Since it is possible, by including the requisite number of fermions, to render b_0 very small, one might hope to construct models for which g_1 is very small. For example if the gauge group is SU(3) and we have 16 or 17 triplets of fermions, then b_0 will equal $1/24\pi^2$ or $-1/24\pi^2$, respectively. This value is suppressed by a factor of roughly 30 compared to the "natural" scale of b_0. Therefore unless there are similar cancellations in the calculation of b_1, we would expect g_1^2 to be rather small, and calculable to a good approximation from the two-loop expression for β. If this is so one could construct models which have UV-stable fixed points at zero or g_1 and IR-stable fixed points at g_1 or zero, respectively. These would provide interesting theoretical models in which both the ultraviolet and the infrared asymptotic behavior would be calculable.

The physical consequences of asymptotic freedom will be explored in the following section and in a subsequent paper. It is clear, from the discussion in Sec. II, that the ultraviolet asymptotic behavior of all Green's functions can be calculated. Thus the n-vector-meson 1PI Green's functions will behave, for large Euclidean momenta, according to Eq. (2.29). If we define (we shall work in the Landau gauge)

$$\gamma_V(g) = c_0 g^2 + c_1 g^4 + \cdots\ , \tag{4.14}$$

where

$$c_0 = -\frac{g^2}{16\pi^2}\left[\tfrac{13}{3}\,C_2(G) - \tfrac{8}{3}\,T(R)\right], \tag{4.15}$$

then

$$\Gamma^{(n)}(\lambda p_1, \ldots, \lambda p_n; g, \mu) \underset{\lambda\to\infty}{\sim} \lambda^{4-n}(\ln\lambda)^{-nc_0/b_0} I_n \times\left[\Gamma^{(n)}(p_1,\ldots,p_n;0,\mu)+\Gamma^{(n)}\left(p_1,\ldots,p_n;\frac{1}{b_0\ln\lambda},\mu\right)+O\left(\frac{\ln\ln\lambda}{\ln\lambda^2}\right)\right], \tag{4.16}$$

where

$$I_n=\exp\left(-n\int_0^t dx[\gamma_V(\bar g(x,g))-c_0\bar g^2]\right)\underset{\lambda\to\infty}{\sim}\text{const}+O\left(\frac{1}{\ln\lambda}\right). \tag{4.17}$$

In particular, the transverse part of the vector-meson propagator behaves like

$$D^{\rm tr}(k)^{ab}_{\mu\nu}\underset{k^2\to-\infty}{\sim}\delta_{ab}\left(g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)(\ln k^2)^{c_0/b_0}I_2. \tag{4.18}$$

To calculate the asymptotic behavior of Green's functions involving fermions we require the anomalous dimension of the fermion field γ_F. This is readily calculated (except for group-theoretical factors it is the same as in QED):

$$\gamma_F(g,\alpha)=g^2 f_0+\cdots$$
$$=\frac{g^2}{16\pi^2}\alpha\, C_2(R)+\cdots. \tag{4.19}$$

Thus the fermion propagator behaves for large momenta according to

$$S_F(p)\underset{p^2\to-\infty}{\sim}\frac{1}{\not p}(\ln p^2)^{f_0/b_0}. \tag{4.20}$$

It is an important feature of asymptotically free theories that the Green's functions are not strictly given by their free-field-theory expression for infinite momenta, due to the presence of the logarithmic term in (4.16). These arise because the anomalous dimensions vanish like $\bar g^2$, and $\bar g^2$ vanishes only logarithmically as $\lambda\to\infty$. This, of course, is a consequence of the fact that $\beta(g^2)$ has a double zero at $g^2=0$. These logarithmic corrections to free-field behavior are calculable and independent of the coupling constants. They can be computed by calculating the relevant anomalous dimension to second order. The above Green's functions are of little physical interest due to their gauge dependence. However, as we shall see shortly, the physically interesting Green's functions of gauge-independent operators (currents in particular) exhibit essentially the same asymptotic properties.

We emphasize that the derivation of the Green's functions in asymptotically free theories does not require the existence of a convergent perturbation theory. It is sufficient to assume that perturbation theory yields an asymptotic expansion, for small coupling constant, of the relevant amplitudes. In that case the renormalization-group equations provide, for asymptotically free theories, a true asymptotic expansion of Green's functions for large Euclidean momenta.

V. MODELS AND APPLICATIONS

In order to construct realistic non-Abelian gauge models of the strong interactions one must confront the issue of symmetry breaking. The standard means of breaking the gauge symmetry is to introduce scalar mesons explicitly into the Lagrangian. As will be explained in Sec. VI it is very difficult to preserve asymptotic freedom while incorporating scalar mesons, and perhaps impossible to include enough scalars to completely break the gauge symmetry. This is not too disappointing since the explicit introduction of scalar mesons, whose only role is to break the symmetry, is not very pleasing. An alternative is that the gauge symmetry is dynamically broken. In other words, a composite Goldstone boson is formed and is eliminated by the usual Higgs mechanism.[40] Another possibility is that *the gauge symmetry is exact*. At first sight this would appear ridiculous since it would imply the existence of massless, strongly coupled vector mesons. However, in asymptotically free theories these naive expectations might be wrong. There may be little connection between the "free" Lagrangian and the spectrum of states.

The possibility of dynamically induced spontaneous symmetry breaking has been considered by many authors,[41,42] although no realistic model has been constructed which does not involve fundamental scalar fields. Of particular interest is the work of Coleman and Weinberg.[42] They show that theories involving massless particles often become unstable, due to infrared singularities, and exhibit spontaneous symmetry breaking. The infrared singularities of a Yang-Mills theory are particularly severe. Furthermore, zero coupling, for such theories, is an ultraviolet-stable fixed point and therefore infrared-unstable. This means that (neglecting masses) as the momenta decrease the effective coupling constant increases. Perturbation theory is therefore totally unreliable insofar as the small-momentum behavior of an asymptotically free theory is concerned. The same renormalization group techniques allow one[42] to discuss the small- or large-classical-field behavior of the "potential." The infrared instability of the origin indicates the unreliability of the classical (free) approximation to this potential. Thus whether or not the theory exhibits symmetry breaking is a difficult dynamical question, requiring nonperturbative calculations.

If the gauge symmetry is broken by a dynamically induced Higgs mechanism then the vector mesons will acquire masses (say 1–3 BeV) and the color degeneracy will be split. In that case one would still be faced with the fact that there is no experimental evidence for the existence of such neutral vector mesons, colored hadrons and especially quarks. The proponents of "red, white, and blue" quarks[43] as a mathematical abstraction argue that the color SU(3) group should be exact, and that all noncolor singlets should be suppressed completely. One clearly requires a dynamical explanation of such a miracle. It might very well be the violent infrared singularities of an asymptotically free gauge theory provide the requisite dynamical mechanism.

To illustrate some of the strange things that could occur in the region of small momenta consider a pure Yang-Mills theory (no fermions). Then the renormalization group equations, Eq. (2.23), as well as the general solution, Eq. (2.27), are exact. One can use them to discuss the on-mass shell, or infrared behavior of the theory by letting $\lambda \to 0$, $t = \ln\lambda \to -\infty$. The effective coupling constant $\bar{g}$, which controls the dynamics in this region is given [in terms of $\bar{g}(t=0,g)=g$] as usual by

$$\int_g^{\bar{g}(t)} \frac{dx}{\beta(x)} = t\,. \tag{5.1}$$

The behavior of $\bar{g}$ as $t \to -\infty$ will depend on the actual form of $\beta(g)$. We can distinguish two cases:

A. $\beta(g)$ vanishes at $g = g_1 < \infty$. In this case $\bar{g}(t)$ will approach g_1 as $t \to -\infty$, its rate of approach will depend on the nature of the zero. This is the simplest case to envisage, the Green's functions will scale according to Eq. (2.29) with some anomalous dimension.

B. $\beta(g)$ is always negative. Here we must further specify whether the integral

$$T = \int_g^{\infty} \frac{dx}{\beta(x)} \tag{5.2}$$

is finite or not.

If $T = \infty$, then $\bar{g}(t)$ approaches infinite values as $t \to -\infty$. Indeed if $\beta(x) \approx x^{\alpha}$, $\alpha < 1$, for large x then $\bar{g}(t) \sim (-t)^{1/(1-\alpha)}$ for large t.

If on the other hand T is finite then the effective coupling constant *diverges for finite momenta*. In fact, $\bar{g}(T,g) = \infty$.

The infrared behavior of Green's functions in this case is determined then by the strong-coupling limit of the theory. It may very well be that this infrared behavior is such so as to suppress all but color singlet states, and that the colored gauge mesons as well as the quarks could be "seen" in the large-Euclidean-momentum region but never be produced as real asymptotic states. This is an exciting possibility which requires further examination.[44]

In any case it might be valid to assume that whatever happens to the theory for small momenta does not affect the ultraviolet behavior. We shall therefore construct models and calculate quantities of physical interest leaving the problem of symmetry breaking (or the lack of it) to further work.

Since we are not to worry about symmetry breaking, our models need not include scalar mesons. We therefore have only to specify the strong-interaction gauge group, G, and the fermion representation R. We would like, of course, to preserve Gell-Mann's (approximate) SU(3)×SU(3) symmetry. This is simply achieved by taking the gauge group to commute with the ordinary SU(3) ×SU(3) generators, and having the fermions belong to a representation of SU(3)×SU(3)×G.[45] We shall take the fermions to be ordinary triplet quarks. The fermions can then be represented by a matrix of spinor fields

$$\Psi = \begin{bmatrix} \mathcal{P}_1 & \mathcal{N}_1 & \lambda_1 \\ \mathcal{P}_2 & \mathcal{N}_2 & \lambda_2 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \mathcal{P}_L & \mathcal{N}_L & \lambda_L \end{bmatrix}. \tag{5.3}$$

The generators of SU(3)×SU(3) transform the columns of this matrix, whereas the generators of G transform its rows. The fermions thus transform as an L-dimensional representation (R) of the gauge group G.

In such a scheme the vector mesons associated with the generators of G are neutral with respect to SU(3)×SU(3). The labels $1, 2, \ldots, L$ which distinguish the different quark triplets can be thought of as colors, so that the strong interactions are mediated by colored gauge mesons. Colored quarks have been considered before[43] for other reasons, and there is some evidence that three colors would be welcome. We shall therefore consider a model in which the strong gauge group is SU(3) and the fermions are color triplets ($L = 3$)

(although as far as asymptotic freedom is concerned any group will do).

In this model Ψ transforms under an infinitesimal gauge transformation according to

$$\Psi(x)\to\Psi(x)+i\epsilon_a(x)\lambda^a\Psi(x) \tag{5.4}$$

and under an ordinary SU(3)×SU(3) transformation according to

$$\Psi(x)\to\Psi(x)+i\Psi(x)\lambda^a\epsilon_a \tag{5.5}$$

[the λ^a are the usual SU(3) matrices]. Our Lagrangian is

$$L=L_V+\mathrm{Tr}\{\bar\psi(i\not\partial-g\not B_a\lambda^a)\psi-\bar\psi\psi M\}, \tag{5.6}$$

where L_V is given by Eq. (2.5) and M is the fermion mass matrix. This model is asymptotically free. The numerical value of $\beta(g)$ is

$$\beta(g)=-\frac{b_0}{2}g^3+\cdots$$
$$=-\frac{g^3}{2}\left(\frac{9}{8\pi^2}\right)+\cdots. \tag{5.7}$$

The ordinary SU(3)×SU(3) vector, V_μ^a, and axial-vector, A_μ^a, currents are given by

$$V_\mu^a=\mathrm{Tr}\{\bar\Psi\gamma_\mu\psi\lambda^a\},$$
$$A_\mu^a=\mathrm{Tr}\{\bar\Psi\gamma_\mu\gamma_5\psi\lambda^a\}. \tag{5.8}$$

Let us now discuss deep-inelastic scattering in such an asymptotically free theory. Due to the fact that the vector mesons are neutral and that the theory is asymptotically free one can derive all the sum rules and relations previously derived in the parton or light-cone models. In addition there will be logarithmic deviations from Bjorken scaling which can be calculated. The full analysis will be given in a subsequent paper[30]; here we shall give a simplified discussion.

The structure functions of deep-inelastic scattering are Fourier transforms of the product of electromagnetic currents. In the scaling region one probes this product for lightlike separation of the currents. To discuss this one employs Wilson's operator-product expansion:

$$J(x)J(-x)\underset{x^2\approx 0}{\sim}\sum_n C^{(n)}(x^2,g)\,x^{\mu_1}\cdots x^{\mu_n}O^{(n)}_{\mu_1\cdots\mu_n}(0) \tag{5.9}$$

[where we have suppressed the vector and SU(3) labels on the currents as well as the tensor and SU(3) structure of the operator-product expansion]. The dominant operators in the scaling region are those of twist (equal to dimension minus spin) 2. These are denoted by $O^{(n)}$. The c-number function, $C^{(n)}(x^2,g)$, contains the light-cone singularity and controls the asymptotic behavior of the structure functions. In fact, the moments of the scaling structure functions measure the Fourier transform of $C^{(n)}$:

$$\int_0^1 dx\,x^{n-2}F_i(x,q^2)\underset{-q^2\to\infty}{\sim}\tilde C_i^{(n)}(q^2,g)\langle n|O^{(n)}|n\rangle \tag{5.10}$$

(where x is the standard Bjorken variable, $x=-q^2/2\nu$, and F_i stands for νW_2 or xW_1). Of course the moment is also proportional to the nucleon matrix element of the operator $O^{(n)}$; however, the q^2 dependence is contained in $\tilde C^{(n)}(q^2,g)$.

One can apply renormalization-group techniques to Wilson's expansion to derive an equation relating the dependence of $C^{(n)}$ on x^2 and g.[21–23] If the operator $O^{(n)}$ is multiplicatively renormalizable (renormalization constant Z_n), then one derives

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}+2\gamma_J(g)-\gamma_n(g)\right]\tilde C_i^{(n)}\left(\frac{q^2}{\mu^2},g\right)=0, \tag{5.11}$$

where γ_J is the anomalous dimension of the "current" $J(x)$, and γ_n the anomalous dimension of $O^{(n)}$:

$$\gamma_n(g)=\mu\frac{\partial}{\partial\mu}\ln Z_n\bigg|_{g_u,\alpha_u,\Lambda\text{ fixed}}. \tag{5.12}$$

In the case under consideration the currents (linear combinations of V_μ^a and A_μ^a) are conserved or partially conserved. They therefore have vanishing anomalous dimension, so that the solution of Eq. (5.11) is

$$\tilde C_i^{(n)}\left(\frac{q^2}{\mu^2},g\right)=\tilde C_i^{(n)}(1,\bar g(t,g))\times\exp\left(-\int_0^t\gamma_n(\bar g(x,g))\,dx\right). \tag{5.13}$$

In our asymptotically free models, $\bar g\to 0$ and $\gamma_n(\bar g)$ vanishes as $t\to\infty$. However, $\gamma_n(\bar g)$ does not vanish fast enough to render the integral in Eq. (5.13) convergent. Let

$$\gamma_n(g^2)=\gamma_n g^2+O(g^4); \tag{5.14}$$

then since $\bar g^2\to b_0^{-1}t^{-1}$, we have that

$$\tilde C_i^{(n)}\left(\frac{q^2}{\mu^2},g\right)\underset{-q^2\to\infty}{\sim}\tilde C_i^{(n)}(1,0)(\ln q^2)^{-\gamma_n/2b_0}\times\exp\left(-\int_0^\infty[\gamma_n(\bar g(x,g))-\gamma_n\bar g^2]\,dx\right). \tag{5.15}$$

Therefore the logarithmic deviations from Bjorken scaling are obtainable from the lowest-order calculation of $\gamma_n(g^2)$.

In general a given operator $O^{(n)}$ will not be multiplicatively renormalizable. In particular if there exists more than one operator with the same spin, quantum numbers, and physical dimension then one must take linear combinations of these to obtain operators with definite dimensions. This mixing occurs for the twist-2, SU(3)×SU(3) singlet operators in our theory, since there exist more than one such operator. Thus, for example, for spin 2 both

$$O^{(2)}_{\mu\nu} = \mathrm{Tr}\{\bar\psi\gamma_\mu(i\overleftrightarrow{\partial}_\nu - 2gB^a_\nu\lambda_a)\psi\} + (\mu \leftrightarrow \nu) - (\text{trace terms}) \tag{5.16}$$

and

$$O^{(2)}_{\mu\nu} = F^a_{\mu\alpha}F^\alpha_{a\nu} - \tfrac{1}{4}g_{\mu\nu}F^a_{\alpha\beta}F^{\alpha\beta}_a$$

contribute to the operator-product expansion ($F^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu - igf_{abc}B^b_\mu B^c_\nu$). This mixing and the calculation of the resulting anomalous dimensions will be discussed in a forthcoming publication.[30]

The nonsinglet SU(3)×SU(3) twist-2 operators however are uniquely given in terms of the fermion fields, and have definite dimensions. Let us denote these operators by $O^{(n)a}_{\mu_1\cdots\mu_n}(x,r)$, where r (=±1) denotes the chirality of the operator. They are given by

$$O^{(n)a}_{\mu_1\cdots\mu_n}(x,r) = \frac{i^n}{2n}\sum_{k=1}^{n}\mathrm{Tr}\{\lambda^a\bar\Psi(x)\nabla_{\mu_1}\cdots\nabla_{\mu_{k-1}}\gamma_{\mu_k}\nabla_{\mu_{k+1}}\cdots\nabla_{\mu_n}(1+r\gamma_5)\Psi(x)\} - (\text{trace terms}), \tag{5.17}$$

where $\nabla_\mu = \partial_\mu + igB^a_\mu\lambda^a$ is the covariant derivative. We define $\Gamma_{O^{(n)a}}$ to be an 1PI Green's function with the insertion of the operator $O^{(n)a}_{\mu_1\cdots\mu_n}(x)$. This amplitude will satisfy a renormalization-group equation. In particular if $O^{(n)a}$ is inserted into the fermion two-point function we obtain

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \gamma^a_n(g) - 2\gamma_F(g)\right]\Gamma^{(2)}_{O^{(n)a}} = 0, \tag{5.18}$$

where γ_F is the anomalous dimension of the fermion field. Since β is of order g^3 we may calculate the combination $\gamma^a_n - 2\gamma_F$ to order g^2 by evaluating the logarithmically divergent contributions to $\Gamma^{(2)}_{O^{(n)}}$ to order g^2. These are given by the Feynman graphs of Fig. 8 and yield

$$\gamma^a_n(g) = \gamma^a_n g^2 + \cdots = \frac{g^2}{8\pi^2}C_2(R)\left[1 - \frac{2}{n(n+1)} + 4\sum_{k=2}^{n}\frac{1}{k}\right] + \cdots \tag{5.19}$$

[in the three-triplet model $C_2(R) = \frac{4}{3}$]. As expected the anomalous dimensions are independent of a and r. Having performed this calculation we can now compute the scaling behavior of the nonsinglet pieces of the deep-inelastic structure functions according to Eq. (5.15). If we denote by $N(x,q^2)$ one of these structure functions, say $F_2^{\text{proton}} - F_2^{\text{neutron}}$, then

$$\int_0^1 dx\,x^{n-2}N(x,q^2) \underset{q^2\to-\infty}{\sim} \text{const}(\ln q^2)^{-a_n}\left[1 + O\left(\frac{1}{\ln q^2}\right)\right], \tag{5.20}$$

where

$$A_n = \frac{3C_2(R)}{22C_2(G) - 8T(R)}\left[1 - \frac{2}{n(n+1)} + 4\sum_{k=2}^{n}\frac{1}{k}\right]. \tag{5.21}$$

FIG. 8. The lowest-order correction to $\Gamma^{(2)}_{O^{(n)}}$.

We note that the power of the log is independent of the physical coupling constant and is determined solely by the gauge group (G) and the fermion representation (R). For the three-triplet SU(3) model $A_2 = \frac{16}{81}$, $A_3 = \frac{25}{81}$, and for large n, $A_n \approx \frac{8}{27}\ln(n)$.

In a forthcoming publication we shall analyze lepton-hadron scattering in great detail; meanwhile a few comments are in order.

a. The approach to asymptopia in these theories is logarithmic. All asymptotic relations will be corrected up to terms of order

$$b_0^{-1}\left(\ln\frac{q^2}{\mu^2}\right)^{-1}.$$

The rate of approach to asymptopia depends on the unknown low-momentum behavior of the theory.

b. Bjorken scaling is violated by logarithmic terms, as in Eq. (5.20). These logarithmic corrections are given by second-order perturbation-theory calculations, and depend only on the gauge group and the fermion representation. The only case where such logarithms are absent is when the relevant operator in the Wilson expansion has zero anomalous dimension. This is the case for vector and axial-vector currents as well as the energy-momentum tensor. Thus the first moment of the isoscalar component of the structure functions does scale:

$$\int_0^1 dx\, F_2^{I=0}(x;q^2) \underset{-q^2\to\infty}{\sim} \text{const} + O\left(\frac{1}{\ln q^2}\right), \tag{5.22}$$

whereas the higher moments decrease logarithmically as in Eq. (5.20). Similarly the total neutrino-hadron cross section *will* scale, since it is given by the same (spin-2) moment.

It is an open question whether such a picture is consistent with experiment. The deviations of the asymptotic forms of the moments from exact scaling are quite small, and one would need rather large variations of q^2 to see any marked change. Verification of logarithmic deviations, as well as the numerical values of a_n, would be strong evidence for the above class of theories.

c. Sum rules and other relations between the various structure functions measured in electron or neutrino scattering, which previously were derived from current algebra, the parton model, or a naive light-cone expansion, are all true theorems in our models. This is because, as is seen in Eq. (5.15), the SU(3)×SU(3) and tensor structure of the Wilson expansion will be (for large q^2) that of the free-quark model—the coupling constant effectively vanishing in this region. There is no point in listing these predictions here—they have been reviewed by many authors.[14,15] However, we note that these relations are approached logarithmically. Thus, for example, since the charged constituents have spin $\frac{1}{2}$ (Ref. 16)

$$\frac{\int_0^1 F_L(x,q^2)x^n dx}{\int_0^1 F_T(x,q^2)x^n dx} \underset{q^2\to-\infty}{\sim} f(\ln q^2) + \frac{1}{q^2} g(\ln q^2), \tag{5.23}$$

where

$$f(\ln q^2) \underset{q^2\to-\infty}{\sim} \frac{1}{\ln q^2}.$$

d. Electron-positron annihilation at large energies is controlled by the identity operator in the Wilson expansion of two electromagnetic currents. Since these all have cannonical dimensions the cross section scales, and the coefficient is determined by the free-field limit (s = center-of-mass energy squared)

$$\frac{\sigma_{e^+e^-\to\text{hadrons}}}{\sigma_{e^+e^-\to\mu^+\mu^-}} = \sum Q_i^{\,2}\left[1 + O\left(\left(\ln\frac{s}{\mu^2}\right)^{-1}\right)\right]. \tag{5.24}$$

Finally let us discuss the incorporation of weak and electromagnetic interactions into our models. After all the recent revival of gauge theories was for the purpose of constructing a unified and renormalizable theory of the weak interactions.[46] Is there any problem in combining these theories? Since in our models the strong gauge group commutes with SU(3)×SU(3) one can easily incorporate the weak plus electromagnetic interactions according to any one of the various schemes proposed recently.[46] In fact, as far as the weak interactions are concerned there is no difference between our models and an Abelian vector-gluon interaction.

One would have to go to extraordinarily high energies to ascertain experimentally whether the weak or electromagnetic interactions are asymptotically free $[e^2\ln(q^2/M_W^{\,2})\approx 1]$. If however we assume that these theories are asymptotically free (in which case one is restricted to semisimple gauge groups and one must worry about symmetry breaking as before) then the Baker-Johnson-Adler approach to QED[10] would be unnecessary. The ultraviolet behavior would be controlled by the fixed point at zero coupling.

VI. THE INCORPORATION OF SCALAR PARTICLES

We shall now consider non-Abelian gauge theories which include scalar fields. There are a variety of reasons for considering such theories. One might want to incorporate spin-zero fundamental fields as such, or one might want to employ such fields to provide masses for the vector mesons by means of the Higgs mechanism. Indeed the only known way of breaking the gauge symmetry spontaneously is via scalar mesons which develop nonvanishing vacuum expectation values. We have explored whether one can incorporate a sufficient number of scalar mesons to break the gauge symmetry completely, or at least retain only an Abelian gauge group. In both these cases one would thereby have a theory which is both asymptotically free and which possesses an S matrix in perturbation theory (and not just off-shell Green's functions).

It is well known from the proofs of the renormalization of Higgs theories[35] that their ultraviolet behavior is that of the underlying symmetric theory. In other words the symmetry breaking is a "generalized mass term" and does not affect the asymptotic (Euclidean) behavior of the theory. The argument that this is so is implicit in the proof of renormalizability of Higgs theories,[35] where one shows that the subtractions which render the symmetric theory finite are sufficient to make the asymmetric theory finite.

Let $\Gamma^{(n)}(p_1,\ldots,p_n;v)$ be any Green's function of the asymmetric theory, where v is the vacuum expectation value of a scalar field which breaks the symmetry. $\Gamma^{(n)}$ may be represented as a functional derivative (n times), with respect to the fields, of the generating functional of 1PI's. It is an immediate consequence of this definition that

$$\Gamma^{(n)}(p_1, \ldots, p_n; v) = \sum_0^\infty \frac{v^j}{j} \Gamma^{(n+j)}(p_1, \ldots, p_n; 0, \ldots, 0; 0), \tag{6.1}$$

where on the right-hand side of (6.1) the $\Gamma^{(n+j)}$ are the Green's functions of the symmetric theory with j insertions of the scalar field carrying zero momentum. Now, by simple power counting the 1PI's $\Gamma^{(n+j)}$ are less and less divergent in the ultraviolet region as j increases, so that the ultraviolet behavior of (6.1) is controlled by the first term, i.e., by the ultraviolet behavior of the symmetric theory. Thus the desirable properties of the symmetric theories discussed below will remain intact in the presence of symmetry breaking.

Before we plunge into the analysis an overview is in order. In the case of pure gauge theories asymptotic freedom provides no constraint. When the theories include fermions a weak constraint (not too many fermion multiplets) must be satisfied. On the other hand the requirement of asymptotic freedom will severely constrain gauge theories involving scalar particles. This is because such particles will necessarily have their own self-couplings. One must therefore investigate the asymptotic freedom of these new (dimensionless) coupling constants. It is well known that a scalar field by itself, with a $\lambda\phi^4$ coupling, is not asymptotically free. Therefore the only hope is that the gauge mesons will help render the fixed point $\lambda = 0$ ultraviolet stable. This turns out to be very difficult to achieve—in fact, it is remarkable that it is possible at all. Unfortunately in order to have asymptotic freedom one is forced to large gauge groups and representations. Furthermore we have not been able to find any examples of models in which the gauge symmetry is completely broken. For these reasons the results presented below reinforce our expectation that if asymptotically free theories of the strong interactions are to be sensible, then the symmetry breaking must be dynamical.

Let us now analyze the renormalization-group equations for gauge theories involving scalar mesons. First we note that the scalars have essentially the same effect on $\bar{g}$ as the fermions. In the vicinity of the origin in coupling-constant space, the renormalization-group equation for the gauge coupling $\bar{g}$ does not involve the scalar self-couplings. These would contribute terms to Eq. (4.8) of order $\bar{g}^2\bar{\lambda}^2$, where λ is some scalar self-coupling. However, unless $\lambda \leq O(g^2)$ the scalar self-couplings are not asymptotically free (see below) and thus these corrections are negligible. The net effect of the scalars is to contribute to $\beta(g)$ a term equal to $\frac{1}{8}$ $(\frac{1}{4})$ of the fermion contribution if the scalars transform according to a real (complex) representation of the gauge group. Thus

$$\beta(g) = -\frac{g^3}{16\pi^2}\left[\tfrac{11}{3}C_2(G) - \sum_{\text{fermions}} \frac{4}{3}\frac{C_2(R)d(R)}{r} - \sum_{\text{scalars}} \frac{1}{6}\frac{C_2(R)d(R)}{r}\right] = -\tfrac{1}{2}b_0 g^3 + \cdots \tag{6.2}$$

[where $d(R)$ for the scalars equals the real dimension of the scalar representation R]. There is, therefore, no problem in retaining asymptotic freedom for g, as long as the number of scalars is not too big.

In addition we are now required to study the renormalization-group equations for the scalar couplings. There are, in general, many such couplings. We are, of course, only interested in the quartic couplings. One such coupling exists for any representation of the gauge group, namely the square of the inner product (the quadratic Casimir operator). If the representation in question is real the vertex is given in Fig. 9, and if it is complex the vertex is given in Fig. 10.

A general representation of the gauge group will, in general, possess many additional quartic invariants. For each such invariant there will be perforce a dimensionless coupling constant (some examples will be given below). Therefore, in general, the renormalization-group equations will be a set of coupled nonlinear differential equations. It is useful to consider first a simple example in which there is only one quartic self-coupling. This is the vector representation (real dimension $= 2N$) of SU(N). If ϕ denotes the (complex) scalar field transforming according to this representation, the scalar interaction is given by

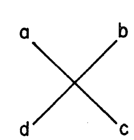
$$L = |(\partial_\mu - igB_\mu)\phi|^2 - \tfrac{1}{2}\lambda(\phi^*\phi)^2. \tag{6.3}$$

The renormalization-group equation for λ will now be

$$\frac{d\bar{\lambda}}{dt} = \beta_\lambda(\bar{\lambda}, \bar{g}), \quad \bar{\lambda}(t=0, \lambda) = \lambda, \tag{6.4}$$

where

$$\beta_\lambda = \mu\frac{\partial\lambda}{\partial\mu}\bigg|_{\lambda_u, g_u, \alpha_u, \Lambda \text{ fixed}}. \tag{6.5}$$



$$-i\lambda\,(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})$$

FIG. 9. The canonical quartic scalar coupling for real representations.

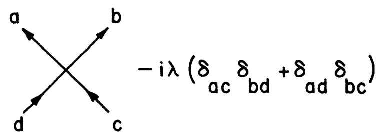


FIG. 10. The canonical quartic scalar coupling for complex representations.

Once again it is sufficient to calculate the logarithmically divergent corrections, in the one-loop approximation, to the quartic scalar vertex. These are illustrated in Fig. 10. [The graphs in Fig. 11(c) should be divided by 2 since it is the square root of the wave-function renormalization constant that enters the renormalization-group equations.] The terms illustrated in Figs. 11(a)–11(d) contribute to β_λ terms of order λ^2, λg^2, λg^2, and g^4, respectively. We therefore obtain an equation for the effective coupling constant $\bar{\lambda}$:

$$\frac{d\bar{\lambda}}{dt}=A\bar{\lambda}^2+B'\bar{\lambda}\bar{g}^2+C\bar{g}^4\ . \tag{6.6}$$

The values of A, B', and C are readily calculated (they are gauge-independent, but it is simplest to evaluate them in Landau gauge):

$$A=\frac{1}{8\pi^2}(N+4)\,, \tag{6.7}$$

$$B'=-\frac{1}{8\pi^2}\,\frac{3(N^2-1)}{N}\ , \tag{6.8}$$

$$C=\frac{1}{8\pi^2}\,\frac{3(N-1)(N^2+2N-2)}{4N^2}\ . \tag{6.9}$$

Equation (6.6) is most easily analyzed by introducing the parameter

$$\alpha=(\bar{g})^{-2}\bar{\lambda};\quad \alpha(0)=g^{-2}\lambda\ . \tag{6.10}$$

We note that $\bar{\lambda}$ must be of order $\bar{g}^2$, for if it were larger then, since $A>0$, the theory would not be asymptotically free. The coupling α satisfies

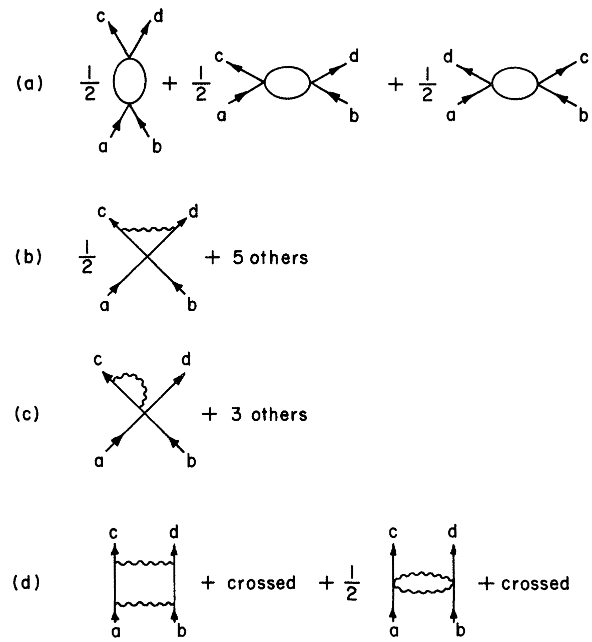


FIG. 11. The graphs that contribute to the β functions for the quartic scalar couplings (the directed lines refer to the complex scalar mesons).

$$\frac{d\alpha}{dt}=\bar{g}^2(A\alpha^2+B\alpha+C)\,, \tag{6.11}$$

where

$$B=B'+b_0\ . \tag{6.12}$$

The fixed points of (6.11) will be given by the zeros of the quadratic form. These will be real and positive if

$$-(4AC)^{1/2}<B<0\ . \tag{6.13}$$

When this condition is satisfied (A and C are always positive) then there will exist two fixed points α_1 and α_2. If the coupling constant $\lambda=\alpha(0)g^2$ is chosen so that

$$\alpha_1=\frac{1}{2A}[-B-(B^2-4AC)^{1/2}]<\alpha(0)<\alpha_2=\frac{1}{2A}[-B+(B^2-4AC)^{1/2}]\ , \tag{6.14}$$

we will be driven to the value $\bar{\alpha}=\alpha_1$ as $t\to\infty$. In other words α_1 is a UV-stable fixed point. Equation (6.11) can easily be solved by quadratures:

$$\int_\alpha^{\bar{\alpha}(t)}\frac{dx}{Ax^2+Bx+C}=\int_0^t dt\frac{g^2}{1+b_0g^2t}\ . \tag{6.15}$$

For large t we have

$$\bar{\alpha}(t)-\alpha_1\underset{t\to\infty}{\sim}t^{-A(\alpha_2-\alpha_1)/b_0}\,, \tag{6.16}$$

and since $\bar{g}^2\approx 1/t$, $\bar{\lambda}(t)$ will approach zero like $1/t$.

It is easy to see that the condition (6.13) will be satisfied, for SU(N), as long as $N\geq 3$ [not SU(2)] and b_0 is small enough. We can always make b_0 small by including fermion representations, whose only effect is to decrease b_0. (We can exclude Yukawa couplings by the discrete symmetry $\phi\to-\phi$.) In the case of SU(3) we must incorporate 16 fermion triplets in order to make b_0 equal to $-1/48\pi^2$.

We see already in this simplest of all cases that the UV stability of λ is extremely delicate. Even in the most favorable case, $N\gg 1$ and $b_0\approx 0$, we can accommodate at most two vector multiplets of scalar fields. Moreover if we do not include enough fermions to render b_0 sufficiently small we cannot include any scalars. Roughly speaking unless b_0 is small the gauge mesons, which are stabilizing the inherently unstable $\lambda\phi^4$ coupling, vanish too rapidly as $t\to\infty$.

Clearly we cannot include enough vector representations to break the gauge symmetry completely. We have also investigated many other models with scalar particles (i.e., other groups and representations). We shall just give a brief summary of the more interesting results.

The adjoint and the symmetric tensor representation of SU(N), as well as the $(N,\bar{N})$ representation of SU(N)×SU(N), are examples of theories with two scalar quartic couplings. They lead to renormalization-group equations of the form (6.6), which can again be simplified by dividing by $\bar{g}^2$ as in (6.11). The result has the form

$$\frac{1}{\bar{g}^2}\frac{d\alpha}{dt}=A^{\alpha}_{\alpha\alpha}\alpha^2+A^{\alpha}_{\alpha\beta}\alpha\beta+A^{\alpha}_{\beta\beta}\beta^2+B_{\alpha}\alpha+C_{\alpha}\,,$$
$$\frac{1}{\bar{g}^2}\frac{d\beta}{dt}=A^{\beta}_{\alpha\alpha}\alpha^2+A^{\beta}_{\alpha\beta}\alpha\beta+A^{\beta}_{\beta\beta}\beta^2+B_{\beta}\beta+C_{\beta}\,. \tag{6.17}$$

The fixed points of these coupled equations are easy to analyze. For each value of α there will be two roots of $d\beta/dt=0$. If these roots are real then the smaller, $\beta_-(\alpha)$, is attractive whereas the larger, $\beta_+(\alpha)$, is repulsive. Similarly we define $\alpha_-(\beta)$ and $\alpha_+(\beta)$. The coupling-constant plane is the pictured in Fig. 12, in the case where all roots are real. We are clearly driven to the fixed point α_f, β_f satisfying

$$\alpha_-(\beta_f)=\alpha_f\,,\qquad \beta_-(\alpha_f)=\beta_f \tag{6.18}$$

as long as the initial values are small enough:

$$\alpha(0)<\alpha_+(\beta(0))\,,\qquad \beta(0)<\beta_+(\alpha(0))\,. \tag{6.19}$$

In practice the simplest way to search for a fixed point is to solve Eq. (6.18) by iteration. We have mapped out some coupling-constant trajectories on a computer, a sample result is given in Fig. 13.

We shall now briefly review the results for the above-mentioned theories. The same Feynman graphs as in the case of the vector representation, Fig. 12, must be evaluated, except that there are now two types of scalar couplings. In each case $\alpha=\lambda/g^2$ and $\beta=\eta/g^2$.

(a) The adjoint representation of SU(N). The scalars are Hermitian traceless $N\times N$ matrices

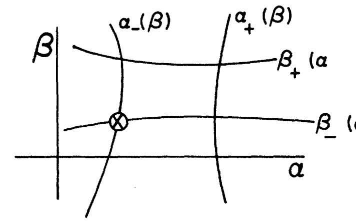


FIG. 12. An illustration of the fixed-point trajectories in the α-β coupling-constant plane. The cross indicates the presence of an ultraviolet-stable fixed point.

M of fields. The quartic self-couplings are given by

$$L=-\tfrac{1}{2}\lambda\,\mathrm{Tr}\{M^2\}-\eta\,\mathrm{Tr}\{M^4\} \tag{6.20}$$

(b) The symmetric tensor representation of SU(N). The scalars are represented by an $N\times N$ symmetric matrix M of (complex) fields. Under the group transformation they transform as $M\to UMU^T$. The quartic scalar interactions are

$$L=-\lambda(\mathrm{Tr}\{MM^\dagger\})^2-\eta\,\mathrm{Tr}\{(MM^\dagger)^2\}\,. \tag{6.21}$$

(c) The $(N,\bar{N})$ representation of SU(N). The scalars are represented by an $N\times N$ matrix M of complex fields. The quartic self-couplings are

$$L=-\tfrac{1}{2}\lambda(\mathrm{Tr}\{MM^\dagger\})^2-\tfrac{1}{2}\eta\mathrm{Tr}\{(MM^\dagger)^2\}. \tag{6.22}$$

The results are shown in Table I. The main result is that all these theories are asymptotically free for N large enough. The UV stability is, as before, delicate. One must make b_0 small, and additional symmetrically coupled scalars destroy the asymptotic freedom.

Our experience in these and other cases suggests that it is the size of a representation, rather than its detailed nature, that determines whether the model will be asymptotically free or not. Indeed in all the cases enumerated in Table I the renormalization-group equations are identical as $N\to\infty$. If one considers larger representations the numberof independent coupling constants rapidly increases and the renormalization-group equations become quite complicated. For example, we have analyzed a theory involving an SU(N)×SU(N) gauge group, with the scalars transforming according to the $(N,\bar{N})\oplus(N,1)\oplus(1,\bar{N})$ representation. In this model there exist five independent scalar coupling constants. It is asymptotically free when $N\geq 5$.

Our original motivation for studying theories involving scalar mesons was to utilize the Higgs mechanism to break the gauge symmetry and provide masses for the vector mesons. Some of the models considered above would appear to have the potential of accomplishing this aim. Take, for example, the gauge group to be SU(N)×SU(N), and

TABLE I. The renormalization-group parameters for two-coupling-constant models. The coefficients refer to Eq. (6.17). The B^γ coefficient is evaluated for $\beta_g = 0$; otherwise B^γ must be modified according to Eq. (6.12).

Representation	γ	$A^\gamma_{\alpha\alpha}$	$A^\gamma_{\alpha\beta}$	$A^\gamma_{\beta\beta}$	B^γ	C^γ	Stability criterion
SU(N): adjoint	α	$\frac{N^2+7}{2}$	$\frac{4N^2-6}{N}$	$\frac{6N^2+18}{N^2}$	$-6N$	9	$N \geq 6$
	β	0	6	$\frac{2N^2-18}{N^2}$	$-6N$	$\frac{3}{2N}$	
SU(N): symmetric tensor	α	$\frac{N(N+1)+8}{2}$	$4(N+1)$	6	$-\frac{6(N^2+N-2)}{N}$	$\frac{9N^2+24}{N^2}$	$N \geq 9$
	β	0	6	$2N+5$	$-\frac{6(N^2+N-2)}{N}$	$\frac{3N^2+12N-48}{2N}$	
SU(N)×SU(N): (N,$\bar{N}$)	α	$\frac{N^2+4}{2}$	$4N$	6	$-\frac{6(N^2-1)}{N}$	$\frac{9N^2+12}{N^2}$	$N \geq 5$
	β	0	6	$2N$	$-\frac{6(N^2-1)}{N}$	$\frac{3N^2-24}{2N}$	

the scalars to transform according to the $(N, \bar{N})$ representation (which is asymptotically free for $N \geq 5$). The scalars are represented by an $N\times N$ matrix M of complex fields, and they transform according to $M \to GMH^\dagger$. If the scalar mesons develop a nonvanishing vacuum expectation value, $\langle 0|M|0\rangle = M_0$, the symmetry will be broken. The criterion that the gauge symmetry be completely broken and that all vector mesons acquire a mass is that there be no subgroup of transformations that leaves M_0 invariant.[47] If M_0 were an arbitrary $N\times N$ complex matrix this criterion would be satisfied. However, M_0 is not arbitrary, rather it is determined by the form of the Lagrangian. Thus in lowest order M_0 is determined by minimizing the "potential"

$$-L_I = -\mu^2 \mathrm{Tr}\{MM^\dagger\} + \lambda\, \mathrm{Tr}\{(MM^\dagger)^2\} + \eta\, \mathrm{Tr}\{(MM^\dagger)^2\}. \tag{6.23}$$

Now $MM^\dagger$ is a positive Hermitian matrix and the potential is a function of the squares of its eigenvalues. When one minimizes (6.23) one finds that these eigenvalues are all equal in magnitude so that $M_0 M_0^\dagger$ is a multiple of the identity. In that case M_0 is, up to a constant, a unitary matrix. There exists an SU(N) subgroup of gauge transformations, $G = G$, $H = M_0 G M_0^{-1}$, that leaves M_0 invariant. Such a model will therefore contain $N^2 - 1$ massive and $N^2 - 1$ massless vector mesons.

In a similar fashion the other two asymptotically free models described above contain massless vector mesons even when the scalars have a nonvanishing vacuum expectation value. Basically the problem is that asymptotic freedom requires large gauge groups and small scalar representations. For such representations the potential in lowest-order perturbation theory, being restricted to renormalizable couplings, is not sufficiently complicated to allow for complete spontaneous symmetry breaking.

We clearly have not exhausted all gauge groups and all scalar representations. One interesting possibility that we have not fully explored is that the symmetry breaking occurs beyond the tree approximation, following Coleman and Weinberg.[42] As we noted above a perturbation-theory calculation of the potential is unreliable for small values of the classical fields (which corresponds to small momenta). However, perturbation theory is reliable for asymptotically free theories, for large values of the classical fields. Therefore there exists the possibility that the potential has a stable minimum for large vacuum expectation values of the scalar fields and that these are calculable using the renormalization group. This possibility is rendered more likely by the fact that in many models the quartic scalar coupling constants (λ and η) can be negative for $p^2 = -\mu^2$ but become positive as $-p^2$ increases, approaching zero from above as $p^2 \to -\infty$ (see Fig. 13). Another possibility is to include Yukawa-like couplings between the scalars and the fermions. In the absence of gauge fields Yukawa couplings are never asymptotically free,[28] but it is easy to see that this is no longer true in gauge theories. We have not explored this possibility in detail.

In conclusion it appears to be very difficult, if not impossible, to construct theories which are both asymptotically free and which contain no massless vector mesons in perturbation theory. Such a model would be of great interest in provid-

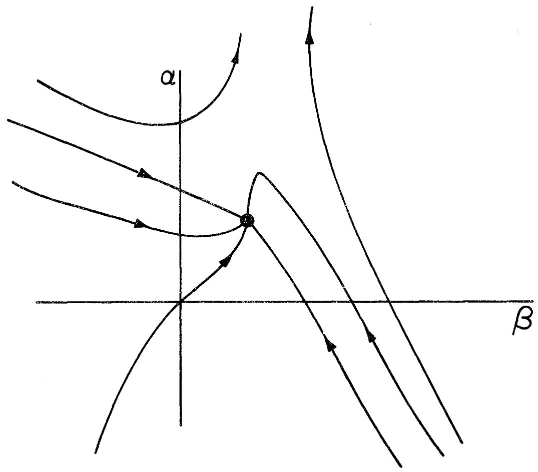


FIG. 13. The coupling-constant trajectories (α and β move along one of the directed lines as t increases). The gauge group is SU(6)×SU(6) and the scalars transform according to the $(6\times\overline{6})$ representation.

ing a theory which possesses an S matrix in perturbation theory and is asymptotically free. However, we do not believe that it would be of physical interest. First, if such a model exists it must be rather complicated and almost unique. Second, as we have seen above, in order to achieve asymptotic freedom it is necessary to incorporate many fermions and render β_g very small. This might imply that β_g has an IR-stable fixed point at a small value of g^2, thus restricting the physical coupling constant $\bar{g}(0,g)$ to be small if we are to remain within the domain of attraction of the origin. Such a model would probably not be useful as a theory of the strong interactions. Furthermore if β_g is small, for small g, then the approach to scaling, according to Eq. (4.16), is very slow. This is clearly not a desirable feature. Therefore, to construct a physically meaningful model of the strong interactions one probably must pin one's hopes on the possibility of dynamical symmetry breaking.

VII. CONCLUSIONS

The theories proposed in this paper are incomplete. The main problem which requires investigation is whether one can obtain an infrared sensible theory without explicit Higgs mesons. One might expect, on physical grounds, that the infrared singularities induced by the gauge charges (color) are so strong that they must be completely shielded, so that only objects neutral under the gauge group could exist. This is an exciting possibility which might provide a mechanism for having a theory of quarks without real quark states. Whether this can be realized or whether the theory will exhibit dynamical symmetry breaking deserves much attention.

What we have achieved so far is to find a large class of asymptotically free theories. We have shown that all semisimple gauge theories are in this class, as well as many theories involving fermions. We have explored the consequences of this asymptotic freedom with respect to deep-inelastic scattering and we have constructed some models which contain scalar mesons. Finally let us recall that the proposed theories appear to be uniquely singled out by nature, if one takes both the SLAC results and the renormalization-group approach to quantum field theory at face value.

APPENDIX A: THE GAUGE DEPENDENCE OF THE RENORMALIZATION-GROUP EQUATIONS

Let us consider the time-ordered product of gauge-invariant operators O_i which are multiplicatively renormalizable. These could be, say, gauge-invariant currents. This renormalized Green's function, which we denote by $G^{(n)}$, will satisfy the renormalization-group equation:

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g,\alpha)\frac{\partial}{\partial g}+\sum_i\gamma_i(g,\alpha)\left(1-2\alpha\frac{\partial}{\partial\alpha}\right)\right]G^{(n)}=0, \tag{A1}$$

where we have used Eq. (2.21) and the sum runs over the anomalous dimensions of the operators O_i.

The derivative with respect to the physical gauge parameter can be eliminated using gauge invariance. This implies that the unrenormalized amplitude $G_u^{(n)}$ is independent of the bare gauge parameter α_u:

$$\frac{\partial}{\partial\alpha_u}G_u^{(n)}(g_u,\Lambda)\bigg|_{g_u,\Lambda\text{ fixed}}=0. \tag{A2}$$

When we recall Eq. (2.14) and express $G_u^{(n)}$ in terms of the renormalized Green's function we have

$$\left[[1-\alpha\sigma(g,\alpha)]\frac{\partial}{\partial\alpha}+\epsilon(g,\alpha)\frac{\partial}{\partial g}+\sum_i\sigma_i(g,\alpha)\right]\times G^{(n)}(g,\alpha,\mu)=0, \tag{A3}$$

where

$$\sigma(g,\alpha)=\frac{\partial Z_3}{\partial\alpha_u}\bigg|_{g_u,\Lambda\text{ fixed}}, \tag{A4}$$

$$\sigma_i(g,\alpha)=Z_3\frac{\partial\ln Z_i}{\partial\alpha_u}\bigg|_{g_u,\Lambda\text{ fixed}}, \tag{A5}$$

$$\epsilon(g,\alpha)=Z_3\frac{\partial g}{\partial\alpha_u}\bigg|_{g_u,\Lambda\text{ fixed}}. \tag{A6}$$

This equation expresses the fact that a change in gauge parameter for gauge-invariant Green's functions can be reabsorbed by a change in coupling constant and in the scale of the operators. It can be used to eliminate the $\partial/\partial\alpha$ in Eq. (A1).

Finally let us note that to lowest order $\beta(g,\alpha)$ is independent of α. This is essentially because, to lowest order, the coupling constant is unique. In other words if we change α we might change g to g'; however, g' (being the value of the three-point function at some point) can be expressed as a power series in g:

$$g' = g + O(g^3). \tag{A7}$$

The term of order g^3 might depend on α; however, to lowest order g' and g must be equal. Therefore

$$\beta'(g') = \mu \frac{d}{d\mu} g' \bigg|_{g_u, \alpha_u, \Lambda \text{ fixed}} = \beta(g) + O(g^5). \tag{A8}$$

Thus the lowest-order term, of order g^3, in $\beta(g)$ must be independent of α.

Similarly the lowest-order term, of order g^2, in the anomalous dimension of a gauge-invariant operator (say those discussed in Sec. V) must be α-independent. This is, of course, not true of the anomalous dimensions of the vector-meson and fermion fields.

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Canonical Estimation of Weak Radiative Corrections in Unified Gauge Theories and Selection Rules*

Rabindra N. Mohapatra, Jogesh C. Pati, and Patrizio Vinciarelli
Department of Physics and Astronomy, Center for Theoretical Physics, University of Maryland, College Park, Maryland 20742
(Received 22 June 1973)

The Bjorken-Johnson-Low limit is used in a canonical current-algebra framework to estimate the magnitude of weak radiative corrections in second and fourth order of the weak coupling constants for unified gauge theories with strong interactions accounted nonperturbatively. We study, in particular, the question of violation of selection rules such as those for parity and strangeness ($|\Delta S|=1$ to second order and $|\Delta S|=2$ to fourth order) for a large class of models, in which strong interactions are generated by Abelian gauges (the neutral-vector-gluon theory) or a special set of non-Abelian gauges.

I. INTRODUCTION

Unified gauge theories of weak and electromagnetic interactions are threatened by the possibility that they may, in general, lead to intolerable magnitudes for the violation of selection rules such as those for parity and strangeness via radiative corrections. The root of the problem rests with that appealing feature of the framework which renders possible the description of a world of interactions having radically different strengths at low energies (momentum transfers) in terms of the same coupling constants ($g, g', \ldots$). Within such a framework a hierarchy of "effective strengths" emerges for the electromagnetic and the weak interactions at low energies in the tree approximation (i.e., $\alpha=g^2/4\pi$ for the electromagnetic and $G_F m^2=g^2m^2/\mu^2$ for the weak; m is some typical hadronic mass and μ is the typical mass of a heavy weak gauge boson) as a consequence of the large difference of mass between the photon and the weak intermediate vector mesons. However, this hierarchy, which also ceases to exist at high energies, is not generally preserved by the weak radiative corrections. The obvious reason is that loop integrations appear which are either divergent or slowly convergent and for which there is not a significant numerical suppression corresponding to the large intermediate boson mass. Such a suppression would indeed only be present where loop integrations converge fast enough to allow one to neglect, to a good approximation, the